Critical behaviour of the non-equilibrium Ising model with locally competing temperatures

1989 J. Phys. A: Math. Gen. 22 4493

(http://iopscience.iop.org/0305-4470/22/20/025)

J. Phys. A: Math. Gen. 22 (1989) 4493-4497. Printed in the UK

COMMENT

Critical behaviour of the non-equilibrium Ising model with locally competing temperatures

M C Marques

Centro de Física da Universidade do Porto (INIC), 4000 Porto, Portugal

Received 9 June 1989

Abstract. We use a method which combines the ideas of the renormalisation group and mean-field theory to obtain the phase diagram of two- and three-dimensional non-equilibrium Ising models with two locally competing temperatures. Comparison is made with available results of other authors. An estimate of critical exponent ν is derived.

Among the simplest examples one can envisage of non-equilibrium systems whose steady states display a phase transition is the Ising model with locally competing temperatures. This is a well defined lattice model system and the non-equilibrium condition results from the competition between two microscopic mechanisms, each one satisfying individually a detailed balance condition at a different temperature.

This model was first considered by Garrido *et al* [1]; these authors did computer simulations and also applied to the two-dimensional system a mean-field-like theory developed for another non-equilibrium problem by Dickman [2]. However, they have not investigated whether the numerical simulation data can indeed be characterised by the classical critical behaviour predicted in their theory or whether the critical exponents for this non-equilibrium model are the same as for the equilibrium Ising model; this is in fact what is argued by Grinstein *et al* [3] should happen for non-equilibrium systems with up–down symmetry, and has already been verified for the Ising model with competing dynamics [4, 5].

In this work we apply the ideas of the mean-field renormalisation group (MFRG), an approach closely related to phenomenological and finite-size scaling which has successfully been applied to equilibrium phase transitions [6]. The method combines the RG assumptions and the idea of mean-field and leads, in general, to better estimates of critical parameters and critical exponents than the ones obtained by mean-field theories. This same approach was recently applied to the non-equilibrium Ising model with competing dynamics [7].

For the model studied here, one spin, picked at random, performs a spin-flip (with probability p $(1-p)$ at the Metropolis rate which corresponds to temperature $\bar{T}$ $(\bar{T}+\overline{\Delta T}))$. The appearance of order in the steady state of this system means that the fraction of spins in state 1 (denoted $f(1)$) is bigger than the fraction of spins in state -1 $(f(-1))$; we write $f(1)=(1+b)/2$, $f(-1)=(1-b)/2$.

We now study a cluster of two spins and consider $P(ij)$, the probability of spins 1, 2 being respectively in states i, j, given that their neighbours outside the cluster have each a probability $(1\pm b)/2$ of being in state ±1. We get for the time evolution of $P(11)$, in the case of a square lattice:

$$\frac{\mathrm{d}P(11)}{\mathrm{d}t}=\frac{A}{2}(P(1-1)+P(-11))-P(11)B \tag{1}$$

where

$$A=\left(\frac{1+b}{2}\right)^3+3\left(\frac{1-b}{2}\right)\left(\frac{1+b}{2}\right)^2+3\left(\frac{1-b}{2}\right)^2\left(\frac{1+b}{2}\right)+\left(\frac{1-b}{2}\right)^3$$
$$\times\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]$$

$$B=\left(\frac{1+b}{2}\right)^3\left[p\exp\left(-\frac{8}{T}\right)+(1-p)\exp\left(-\frac{8}{T+\Delta T}\right)\right]+3\left(\frac{1-b}{2}\right)\left(\frac{1+b}{2}\right)^2$$
$$\times\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]$$
$$+3\left(\frac{1-b}{2}\right)^2\left(\frac{1+b}{2}\right)+\left(\frac{1-b}{2}\right)^3$$

and $T=(k_{\mathrm{B}}\bar{T})/J$, $\Delta T=(k_{\mathrm{B}}\overline{\Delta T})/J$, J being the nearest-neighbour interaction.

The time evolution of $P(-11)+P(1-1)$ and $P(-1-1)$ can be obtained in the same way.

In the vicinity of a second-order phase transition, b is a small quantity; expanding in b, we get, from (1), the stationarity condition

$$P(11)\left\{\left[p\exp\left(-\frac{8}{T}\right)+(1-p)\exp\left(-\frac{8}{T+\Delta T}\right)\right]\right.$$
$$+3\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]$$
$$+4+b\left[3\left[p\exp\left(-\frac{8}{T}\right)+(1-p)\exp\left(-\frac{8}{T+\Delta T}\right)\right]\right.$$
$$\left.\left.+3\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]-6\right]\right\}$$
$$=\frac{P(1-1)+P(-11)}{2}\left\{7+\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]\right.$$
$$\left.+b\left[3-3\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]\right]\right\}+\mathrm{O}(b^2).$$

By considering the other stationary equations, we obtain

$$P(11)-P(-1-1)$$
$$=b\left\{54-18\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]\right.$$
$$-30\left[p\exp\left(-\frac{8}{T}\right)+(1-p)\exp\left(-\frac{8}{T+\Delta T}\right)\right]$$

$$
\begin{aligned}
&-6\left[p\exp\left(-\frac{12}{T}\right)+(1-p)\exp\left(-\frac{12}{T+\Delta T}\right)\right]\bigg\}\\
&\times\bigg\{44+49\left[p\exp\left(-\frac{4}{T}\right)+(1-p)\exp\left(-\frac{4}{T+\Delta T}\right)\right]\\
&+27\left[p\exp\left(-\frac{8}{T}\right)+(1-p)\exp\left(-\frac{8}{T+\Delta T}\right)\right]\\
&+7\left[p\exp\left(-\frac{12}{T}\right)+(1-p)\exp\left(-\frac{12}{T+\Delta T}\right)\right]+p\exp\left(-\frac{16}{T}\right)\\
&+(1-p)\exp\left(-\frac{16}{T+\Delta T}\right)\bigg\}^{-1}+\mathrm{O}(b^2)\\
&=f_{\mathrm{II}}(T,\Delta T,p)b+\mathrm{O}(b^2).
\end{aligned}
$$

Considering now a one-spin cluster, we denote by $P(i)$ the probability of this spin being in state i, given that its neighbours have each a probability $(1\pm b')/2$ of being in state ±1.

The stationary condition is, in this case

$$
\begin{aligned}
&P(1)-P(-1)\\
&\quad=b'\bigg\{12-8\left[p\exp\left(-\frac{4}{T'}\right)+(1-p)\exp\left(-\frac{4}{T'+\Delta T}\right)\right]\\
&\quad-4\left[p\exp\left(-\frac{8}{T'}\right)+(1-p)\exp\left(-\frac{8}{T'+\Delta T}\right)\right]\bigg\}\\
&\quad\times\bigg\{11+4\left[p\exp\left(-\frac{4}{T'}\right)+(1-p)\exp\left(-\frac{4}{T'+\Delta T}\right)\right]\\
&\quad+p\exp\left(-\frac{8}{T'}\right)+(1-p)\exp\left(-\frac{8}{T'+\Delta T}\right)\bigg\}^{-1}+\mathrm{O}(b'^2)\\
&\quad=f_{\mathrm{I}}(T',\Delta T,p)b'+\mathrm{O}(b'^2).
\end{aligned}
$$

The important assumption of MFRG is to consider that $P(1)-P(-1)$ and $P(11)-P(-1-1)$ must, in the vicinity of the transition, scale like b' and b, respectively. This gives the RG recursion relation $K'=K'(K)$ (where $K'=1/T'$, $K=1/T$), and the fixed point equation

$$
f_{\mathrm{I}}\left(\frac{1}{K_{\mathrm{c}}},\Delta T,p\right)=f_{\mathrm{II}}\left(\frac{1}{K_{\mathrm{c}}},\Delta T,p\right). \tag{2}
$$

In figures 1 and 2 we have represented the variation of T_{c} with p and ΔT as obtained from (2) and compared the present results with those of [1] where Dickman's method was used. A rather good agreement between the two theories is displayed. As concerns the value p^*, above which T_{c} never reaches zero regardless of the magnitude of ΔT, our estimate is $p^*=0.910$, slightly above the value $p^*=\frac{27}{32}$ obtained in [1]. Figure 3 shows the equivalent plot for the three-dimensional system; we get in that case $p^*=0.900$.

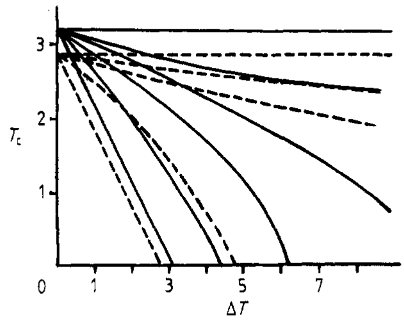

Figure 1. Plots of T_c as a function of ΔT for $d=2$ and different values of p. The full curves represent MFRG results for $p=0$, 0.5, 0.7, 0.8, 0.9, 1.0 (from bottom to top) and the broken curves represent results of [1] for $p=0$, 0.5, 0.844, 0.9, 1.0.

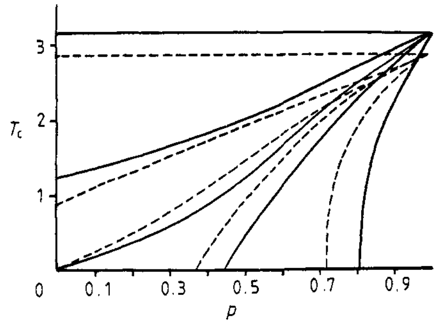

Figure 2. Plots of T_c as a function of p for $d=2$ and different values of ΔT. The full curves represent MFRG results for $\Delta T=0$, 2, 3.14, 4, 10 (from top to bottom) and the broken curves represent results of [1] for $\Delta T=0$, 2, 2.88, 4, 10.

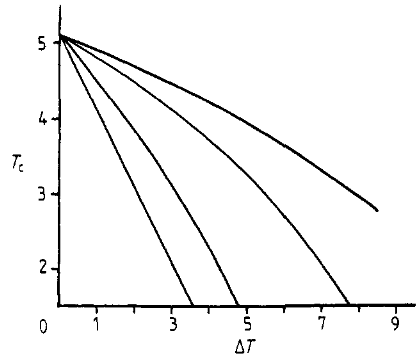

Figure 3. Plot of T_c as a function of ΔT for $d=3$ and $p=0$, 0.5, 0.7, 0.8, as obtained from MFRG.

There does not seem to exist enough numerical simulations to test the phase diagram for the entire parameter space. Garrido *et al* [1] report a far smaller variation of the critical temperature when ΔT varies from $\Delta T=0.1$ to $\Delta T=1.0$ at $p=0.5$ than the approximate 15% decrease given by mean-field and our method. However, their MC results do not extend to higher values of ΔT or different p.

In what concerns critical exponents it still remains to clarify whether they are the classical ones predicted by Garrido *et al* [1] or whether the argument of Grinstein *et al* [3] holds and they are in fact the equilibrium Ising exponents.

The application of the MFRG method to equilibrium systems leads usually to better estimates for the critical parameters than for the critical exponent, at least when comparison of very small clusters is involved; however slow the convergence to the exact values may be, they still represent a better approximation than mean-field theory.

We have estimated the critical exponent ν by linearisation of the recursion relation around the fixed point

$$\left.\frac{\partial K'(\Delta T, p)}{\partial K(\Delta T, p)}\right|_{K_c(\Delta T, p)} = l^{1/\nu}$$

where l is the scaling factor.

For $d = 2$, we used $l = \sqrt{2}$, and obtained $\nu = 1.2$ for $\Delta T = 3.0$, $p = 0.4$, with a variation of a few percent when the parameters ΔT and p were changed. This compares well with the exact value for the equilibrium Ising model, $\nu = 1$, and is within the level of approximation that MFRG attains in equilibrium systems when clusters of small size are considered.

For $d = 3$, we used a scaling factor corrected for the lack of appropriate symmetry of the two clusters involved; according to Slotte [8], l should in this case be $l = 3\sqrt{\frac{2}{13}}$. This gives, within the present approximation, $\nu = 0.62 \pm 0.02$, which compares very well with the series expansions result for the three-dimensional equilibrium Ising system $\nu = 0.63$.

We think that the consideration of bigger clusters, namely of more appropriate symmetry, is likely to improve the results. However, we are able by this present approach to predict for the two-dimensional system a phase diagram in close agreement with the one obtained by a different method, and extend our results to the three-dimensional case; our estimates of ν, although not very accurate, point in the direction of a critical behaviour characterised by the equilibrium Ising exponents.

References

[1] Garrido P L, Labarta A and Marro J 1987 *J. Stat. Phys.* **49** 551
[2] Dickman R 1987 *Phys. Lett.* **122A** 463
[3] Grinstein G, Jayaprakash C and He Yu 1985 *Phys. Rev. Lett.* **55** 2527
[4] Gonzalez-Miranda J M, Garrido P L, Marro J and Lebowitz J L 1987 *Phys. Rev. Lett.* **59** 1934
[5] Wang Jian-Sheng and Lebowitz J L 1988 *J. Stat. Phys.* **51** 893
[6] Indekeu J O, Maritan A and Stella A L 1987 *Phys. Rev.* B **35** 305
[7] Marques M C 1989 *Preprint*
[8] Slotte P A 1987 *J. Phys. A: Math. Gen.* **20** L177